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Undirected simple connected graphs with minimum number of spanning trees

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ABSTRACT

We show that for positive integers n, m with $n(n-1)/2 \geq m \geq n-1$, the graph $L_{n,m}$ having n vertices and m edges that consists of an $(n-k)$-clique and $k-1$ vertices of degree 1 has the fewest spanning trees among all connected graphs on n vertices and m edges. This proves Boesch's conjecture [F.T. Boesch, A. Satyanarayana, C.L. Suffel, Least reliable networks and reliability domination, IEEE Trans. Commun. 38 (1990) 2004–2009].

1. Introduction

Let $t(G)$ denote the number of spanning trees in the connected simple undirected graph G. Given positive integers n and m for which there are connected graphs on n vertices and m edges, it is natural to try to determine which graphs maximize or minimize $t(G)$, when G ranges over all connected graphs on n vertices and m edges.

It turns out the maximization version is more difficult and only special cases have been resolved to date [2,8,13]. The minimization problem has been attacked with rather more success [1,4,9]. Boesch conjectured that, for positive integers n and m for which there are connected graphs with n vertices and m edges, a particular graph (described below) minimizes the number of spanning trees [1]. In particular, Kelmans et al. proved the conjecture if $m \geq n(n-1)/2 - n + 2$, in which case $L_{n,m}$ consists of an $(n-1)$—clique and one vertex joined to at least one of the vertices of the clique [9].

In this paper we prove Boesch's Conjecture. To obtain the graph $L_{n,m}$, let k be the least integer such that $m \geq (n-k)(n-k-1)/2 + k$. Then $L_{n,m}$ consists of $(n-k)$—clique, joined to $k-1$ vertices of degree 1, plus one other vertex of degree $m-(n-k)(n-k-1)/2-k-1$, joined to vertices of the clique. We shall follow the terminology and notation of the book by Harary [6].

2. Shifting transformation

The first step in our proof of Boesch's Conjecture is to employ Kelmans' shifting transformation on undirected graphs [7,12]. Let $G = (V, E)$ be an undirected simple graph and, for a vertex v of G, let $N(v)$ denote the vertices that are neighbors to v. The graph shift(G, v, w) is obtained from G by, for all $x \in N(v) \setminus (N(w) \cup \{w\})$ deleting vx and adding wx. The following is known [3,4].

Lemma 2.1. *For any connected graph G and any vertices v, w of G,*

$$t(\text{shift}(G, v, w)) \leq t(G).$$

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Furthermore, it is known that if shift$(G, v, w) = G$, then G is a threshold graph [1,3,4,11]. These are the graphs $H = H(n; d_1, d_2, \ldots, d_k)$ consisting of $(n-k)-$clique, with vertices $v_{k+1}, v_{k+2}, \ldots, v_n$, and an independent set on the remaining k vertices, the ith one of which is joined to $v_{k+1}, v_{k+2}, \ldots, v_{k+d_i}$.

It was shown in [1,3,4] that every simple connected graph G can be transformed into a threshold graph H using a series of shift(G, v, w) transformations. Consequently:

Theorem 2.2. *For any connected graph G, there is a threshold graph H, with the same numbers of vertices and edges, such that* $t(H) \leq t(G)$.

Thus, the second step in the proof will be to determine the number of spanning trees in $H(n; d_1, d_2, \ldots, d_k)$, which is done in the next section. Recall that a vertex v *dominates* a vertex w if $N(w) \setminus \{v\} \subseteq N(v) \setminus w$. In $H(n; d_1, d_2, \ldots, d_k)$, the vertices may be ordered $v_1, v_2, \ldots, v_k$ so that, if $i < j$, then v_i dominates v_j. This will be useful in determining $t(H)$.

3. The number of spanning trees in *H*

In this section we prove the following result for $H = H(n; d_1, d_2, \ldots, d_k)$.

Theorem 3.1. *Suppose* $H = H(n; d_1, d_2, \ldots, d_k)$ *is a connected graph, with* $d_1 \geq d_2 \cdots \geq d_k$. *Set* $d_0 = n-k$ *and* $d_{k+1} = 1$.*Then*

$$t(H) = (n-k)^{-2} \prod_{i=0}^{k} (d_i(n-k+i)^{d_i - d_{i+1}}). \tag{1}$$

A classic result of Kirchoff also known as Matrix Tree Theorem [5] can be used to calculate $t(G)$ for any graph G. Let A be the adjacency matrix of G and let D be the diagonal matrix whose diagonal entries are the degrees of vertices of G using the same indexing of rows and columns in both A and D. The Matrix Tree Theorem asserts that the number of spanning trees of G is the determinant of any of the principal $(n-1) \times (n-1)$ submatrices of $D - A$.

To establish the principal $(n-1) \times (n-1)$ submatrix of $D - A$ for H we use the following labeling of the vertices. For $1 \leq i \leq k$, v_i has degree d_i, and v_i is adjacent only to vertices $v_{k+1}, v_{k+2}, \ldots, v_{k+d_i}$. For $k < i \leq k + d_k$, v_i has degree $n-1$, and v_i is adjacent to all vertices. For $k + d_k < i \leq k + d_1$, v_i has degree $d_i \leq d_{i-1}$, and if $d_i = d_{i-1}$ then v_i is adjacent to the same vertices as v_{i-1}. Otherwise, v_i has degree $n-k-1+r < d_{i-1}$ for some integer $r \geq 1$, v_i is adjacent to vertices $v_1, v_2, \ldots, v_r$ and v_i is also adjacent to each vertex v_j, where $j > k$ and $j \neq i$. For $i > k + d_1$, v_i has degree $n-k-1$, and v_i is adjacent to each vertex v_j, where $j > k$ and $j \neq i$.

To state the result for $t(H)$ we form the Kirchoff matrix $D - A = A_n$ based on the above vertex labeling, where row i corresponds to vertex v_{n-i+1} and column j corresponds to vertex v_{n-j+1}. We now focus attention on the principal $(n-1) \times (n-1)$ submatrix of A_n, obtained by deleting its row and column corresponding to vertex v_k.

The principal submatrix A_{n-1} is shown in Fig. 1. In the following proof of Theorem 3.1 we will evaluate the determinants in three main steps. First we will reduce the computation to the computation of a determinant D_1. Then we will derive the recursion for D_i in terms of D_{i+1}, and finally we will determine D_k. The columns will be denoted by $c_1, c_2, \ldots, c_i$ and the rows will be denoted by $r_1, r_2, \ldots, r_i$.

Proof of Theorem 3.1. For $k = 0$, $H = K_n$ and (1) is satisfied. For $k = 1$, H represents a complete graph with removed star. The formula for $t(H)$ in this case can be found in [9] that also satisfies (1). Hence, without loss of generality we consider H for $k \geq 2$. Clearly, $n - d_1 - k \geq 0$ must be satisfied. If $n - d_1 - k > 0$, then we first evaluate $\det(A_{n-1})$ through the following steps 1–3. Otherwise we skip these three steps.

1. Subtract last column c_{n-1} from columns $c_1, c_2, \ldots, c_{n-d_1-k}$.
2. Add rows $r_1, r_2, \ldots, r_{n-d_1-k}$ to the last row r_{n-1}.
3. Subtract column c_{n-d_1-k} from columns $c_1, c_2, \ldots, c_{n-d_1-k-1}$,
and then add rows $r_1, r_2, \ldots, r_{n-d_1-k-1}$ to row r_{n-d_1-k}.

After further factoring out the vertices of degree $n-k-1$ we get

$$t(H) = d_1(n-k)^{n-d_1-k-1} D_1, \tag{2}$$

where D_i for $i \geq 1$ is represented in Fig. 2. We can now verify that for case $n - d_1 - k = 0$ we have $d_1(n-k)^{n-d_1-k-1} = 1$ and $\det(A_{n-1}) = D_1$.

In the following steps 4–9 we derive recursion for D_i, for $i \leq k-2$.

4. Subtract the last column c_{d_i+k-i} from columns $c_1, c_2, \ldots, c_{d_i-d_{i+1}}$.
5. Add rows $r_1, r_2, \ldots, r_{d_i-d_{i+1}}$ to the last row r_{d_i+k-i}.
6. Reduce D_i by eliminating first $d_i - d_{i+1}$ rows and columns from D_i (Fig. 3).
7. Subtract column $c_{d_{i+1}+k-i-1}$ from the last column $c_{d_{i+1}+k-i}$.
8. Add row $r_{d_{i+1}+k-i-1}$ to the last row $r_{d_{i+1}+k-i}$.
9. Expand D_i with respect to the last column.

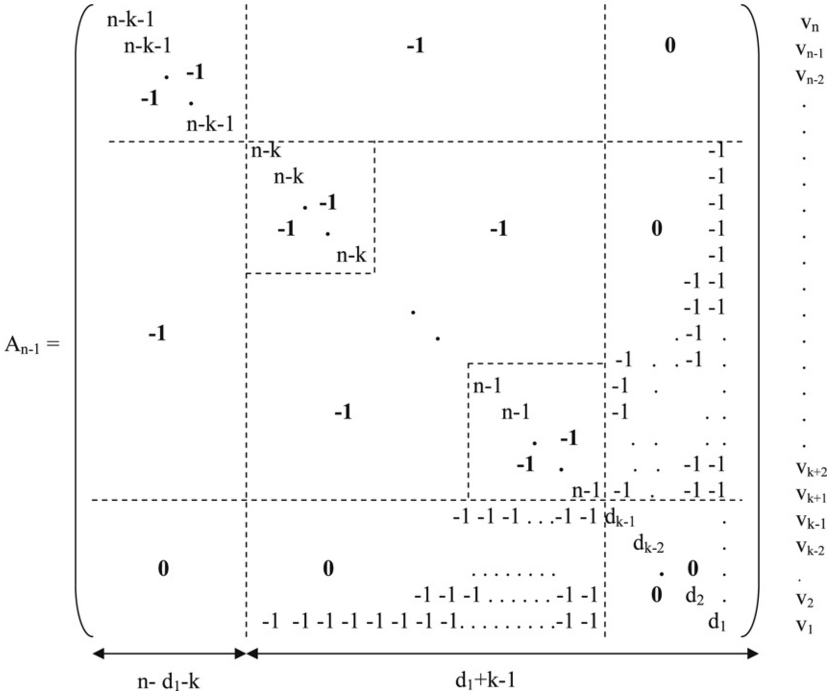

Fig. 1. Matrix A_{n-1}.

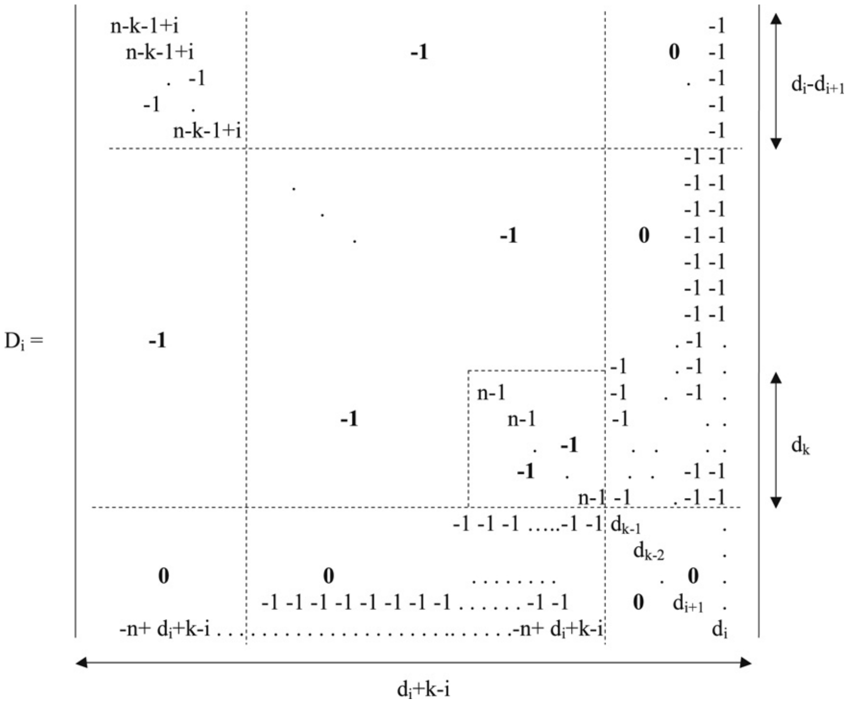

Fig. 2. D_i for $i \geq 1$.

Thus, we obtain the following recursion:

$$D_i = (n - k + i)^{d_i - d_{i+1}} d_{i+1} D_{i+1}. \tag{3}$$

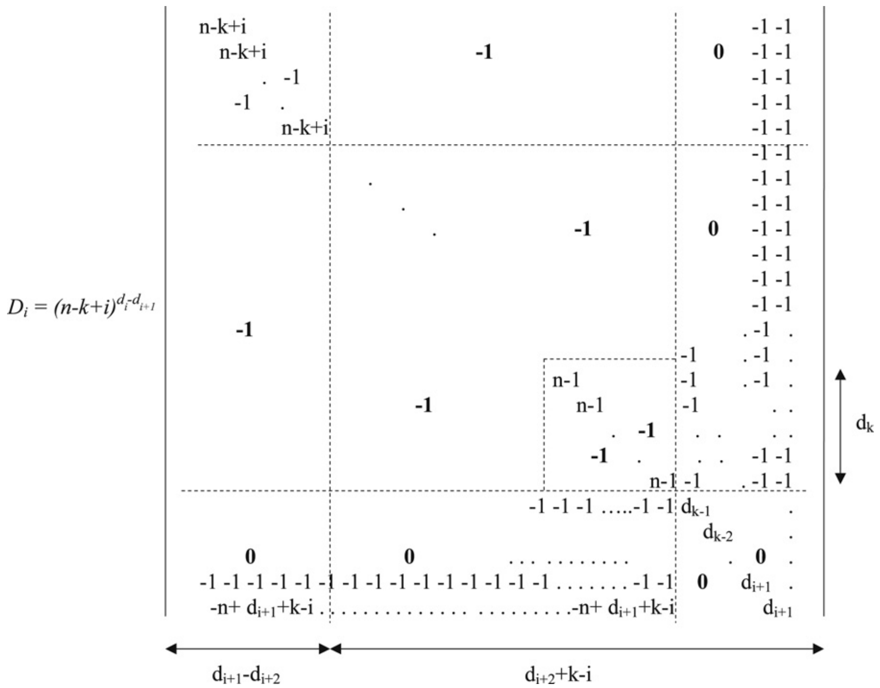

Fig. 3. Evaluation of D_i after step 6.

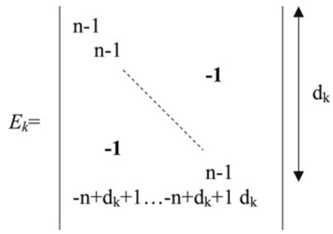

Fig. 4. E_k.

Hence, D_1 can be expressed by

$$D_1 = D_{k-1} \prod_{i=1}^{k-2} (d_{i+1}(n-k+i)^{d_i - d_{i+1}}). \tag{4}$$

We evaluate D_{k-1} through steps 10–12 as follows:

10. Subtract the last column $c_{d_{k-1}+1}$ from columns $c_1, c_2, \ldots, c_{d_{k-1}-d_k}$.

11. Add rows $r_1, r_2, \ldots, r_{d_{k-1}-d_k}$ to the last row $r_{d_{k-1}+1}$.

12. Reduce D_{k-1} by eliminating first $d_{k-1} - d_k$ rows and columns from D_{k-1}.

So, D_{k-1} can be expressed as

$$D_{k-1} = (n-1)^{d_{k-1}-d_k} E_k \tag{5}$$

where E_k is illustrated in Fig. 4.

Subsequently, we evaluate E_k through steps 13–16 as follows:

13. Subtract the last column c_{d_k+1} from columns $c_1, c_2, \ldots, c_{d_k}$.

14. Add rows $r_1, r_2, \ldots, r_{d_k}$ to the last row c_{d_k+1}.

15. Factor out $1/n$ from the last column c_{d_k+1}.

16. Add columns $c_1, c_2, \ldots, c_{d_k}$ to the last column c_{d_k+1}.

We obtain

$$E_k = n^{d_k - 1} d_k. \tag{6}$$

Hence, after inserting (6) into (5), then (5) into (4), and finally (4) inserting into (2) we get

$$t(H) = d_1(n-k)^{n-d_1-k-1} n^{d_k-1} d_k (n-1)^{d_{k-1}-d_k} \prod_{i=1}^{k-2} (d_{i+1}(n-k+i)^{d_i-d_{i+1}}), \tag{7}$$

which equals (1) for $d_0 = n-k$ and $d_{k+1} = 1$. $\square$

4. Main result

In the third step we focus on the threshold family of graphs. We derive properties for $H = H(n; n-k, \dots, n-k, d_i, 1, \dots, 1)$ based on the corresponding function $f(x_1, x_2, \dots, x_k)$ in Lemma 4.1 through 4.4 [10].

Lemma 4.1. *Let b, c, k, be given positive integers with $b \geq 3$ and $kb-k \geq c > k$. Let $x_0 = b$, $x_{k+1} = 1$, and let $f(x_1, x_2, \dots, x_k) = \prod_{i=0}^{k}(x_i(b+i)^{x_i - x_{i+1}})$. The minimum of f over the region*

$$P := \left\{ x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1 \right\}$$

occurs at some point $(x_1, x_2, \dots, x_k)$ that satisfies at most two of the following inequalities strictly:

$$b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1.$$

Proof. Since P is a nonempty polytope and f is continuous over P, the desired minimum exists and is attained in P. $f : P \to \mathbb{R}$ takes only positive values. So, $F : P \to \mathbb{R}$,

$$F(x) := \ln(f(x))$$

is well defined. Since ln(.) is strictly monotone, the original optimization problem is equivalent to

$$\min\{F(x) : x \in P\}.$$

The latter has the same set of optimal solutions as the problem

$$\max\{-F(x) : x \in P\}.$$

We compute

$$F(x) = \sum_{i=1}^{k} [\ln(x_i) + x_i \ln(1 + 1/(b+i-1))] + \text{constant}.$$

The Hessian of $-F$ is the diagonal matrix

$$\begin{pmatrix} x_1^{-2} & & & \\ & x_2^{-2} & & \mathbf{0} \\ & & \cdot & \\ \mathbf{0} & & & \cdot \\ & & & & x_k^{-2} \end{pmatrix}.$$

Thus, the Hessian is positive definite over P and hence $-F$ is *strictly convex* over P. Therefore, every optimal solution must be an extreme point of P. Using the linear algebraic characterization of extreme points of polytopes on P, we conclude that the minimum value of f over P is finite, and every minimizer x satisfies at most two of the following inequalities strictly (all others are satisfied with equality):

$$b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1. \quad \square$$

Lemma 4.2. *Let b, c, k, be given positive integers with $b \geq 3$ and $kb-k \geq c > k \geq 2$. Let u be given nonnegative integer. Let $x_0 = b$, $x_{k+1} = 1$, and let $f(x_1, x_2, \dots, x_k) = \prod_{i=0}^{k}(x_i(b+u+i)^{x_i - x_{i+1}})$. Let $f_1 = f(x_1, x_2, \dots, x_k)$ if $x_1 = x_2 = \cdots = x_k = c/k$, and let $f_2 = f(x_1, x_2, \dots, x_k)$ if $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r \geq 1$ and $x_{r+1} = x_{r+2} = \cdots = x_k = 1$, for $r \geq 1$. Then $f_1 > f_2$ is satisfied over the region*

$$P := \left\{ x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1 \right\}.$$

Proof. We define functions $g_1(b,c,k,u)$, $g_2(b,c,k,u,r)$ corresponding to f_1, f_2 respectively as follows:

$$g_1(b,c,k,u) = b(c/k)^k(b+u)^{b-\frac{c}{k}}(b+u+k)^{\frac{c}{k}-1} \tag{8}$$

and

$$g_2(b,c,k,u,r) = b^r(c+1-(b-1)(r-1)-k)(b+u+r-1)^{b-(c+1-(b-1)(r-1)-k)}(b+u+r)^{c-(b-1)(r-1)-k}. \tag{9}$$

For the purpose of this evaluation we assume $b, c, k, u, r \in R$. The proof follows by direct comparison of $g_1(b,c,k,u)$ with $g_2(b,c,k,u,r)$.

We first compare $g_1(b,c,k,u)$ with $g_2(b,c,k,u,r)$ for $c = k+1$ (least possible) and for given b, k, u. Then,

$$g_1(b,k,u) = b((k+1)/k)^k(b+u)^{(kb-k-1)/k}(b+u+k)^{1/k} \tag{10}$$

and

$$g_2(b,u) = 2b(b+u)^{b-2}(b+u+1). \tag{11}$$

Define $g_3(b,k,u) = \ln(g_1(b,k,u)/g_2(b,u))$. Then

$$\begin{aligned} k\partial g_3(b,k,u)/\partial u &= (k-1)/(b+u) + 1/(b+u+k) - k/(b+u+1) \\ &= (k^2-k)/((b+u)(b+u+1)(b+u+k)) > 0. \end{aligned}$$

So, because $\partial g_3(b,k,u)/\partial u > 0$ then without loss of generality we assume $u = 0$, and we compare

$$g_1(b,k) = ((k+1)/k)^k b^{(kb-1)/k}(b+k)^{1/k}$$

with

$$g_2(b) = 2b^{b-1}(b+1).$$

By direct calculation we have

$$k\partial(\ln(g_1(b,k)/g_2(b)))/\partial b = (k-1)/b + 1/(b+k) - k/(b+1) = (k^2-k)/(b(b+k)(b+1)) > 0.$$

So, our examination simplifies to the following

$$((k+1)/k)^{k^2}3^{k-1}(3+k) > 8^k. \tag{12}$$

For $26 \geq k \geq 2$ we numerically verified that (12) holds. For $k \geq 26$ we evaluate it as follows:

$$((k+1)/k)^{k^2}3^{k-1}(3+k) > ((k+1)/k)^{k^2}3^k.$$

We verify that $((k+1)/k)^{k^2}3^k > 8^k$ holds for $k = 26$. Inequality $((k+1)/k)^{k^2}3^k > 8^k$ is equivalent to $((k+1)/k)^k > 8/3$. Let $G(k) = \ln(((k+1)/k)^k/(8/3))$. Then,

$$\begin{aligned} \mathrm{d}(G(k))/\mathrm{d}k &= \ln((k+1)/k) - 1/(k+1) \\ &= [1/(k+1) + (1/2)(1/(k+1))^2 + (1/3)(1/(k+1))^3 + \cdots] - 1/(k+1) > 0. \end{aligned}$$

So, $((k+1)/k)^k > 8/3$ holds for $k \geq 26$, which implies that (12) also holds. Hence, we conclude that $g_1(b,c,k,u) > g_2(b,c,k,u,r)$ for $c = k+1$.

We now compare $g_1(b,c,k,u)$ with $g_2(b,c,k,u,r)$ for $c = kb-k$ (largest possible), and for given $b \geq 3$, $k \geq 2$, $u \geq 0$. In order to establish r in (9) for comparison, we introduce a substitution $k = k' + (b-1)w$, where $b-1 \geq k' \geq 1$ and $w \geq 0$. Then,

$$g_1(b,u,k',w) = b(b+u)(b-1)^{k'+(b-1)w}(b+u+k'+(b-1)w)^{b-2} \tag{13}$$

and

$$\begin{aligned} g_2(b,u,k',w) &= b^{(k'+(b-1)w)-w}(b-k')(b+u+(k'+(b-1)w)-w-1)^{b-(b-k')} \\ &\quad \times (b+u+(k'+(b-1)w)-w)^{b-k'-1} \\ &= b^{k'+(b-2)w}(b-k')(b+u+k'+(b-2)w-1)^{k'}(b+u+k'+(b-2)w)^{b-k'-1}. \end{aligned} \tag{14}$$

Define $g_3(b,u,k',w) = \ln(g_1(b,u,k',w)/g_2(b,u,k',w))$. For $b = 3$ we evaluate $\partial g_3(b,u,k',w)/\partial w$ for the points where w becomes integer. Based on the straightforward evaluation, which we leave here to the reader, we obtain the following:

$$\partial g_3(b,u,k',w)/\partial w \geq 2\ln(2) - \ln(3) - 2(1/(4+w-1) - 1/(4+2w)).$$

For $w \geq 0$ expression $1/(4+w-1) - 1/(4+2w)$ has maxima for integers $w = 0$ and $w = 1$ with minimum $\partial g_3(b, u, k', w)/\partial w = 2\ln(2) - \ln(3) - 2/3 + 1/2 > 0$. For $w > 1$, $\partial g_3(b, u, k', w)/\partial w > \partial g_3(b, u, k', w)/\partial w|_{w=1} > 0$. For $b \geq 4$ we evaluate $\partial g_3(b, u, k', w)/\partial w$ in straightforward way (again we leave it to the reader) and obtain the following:

$$\partial g_3(b, u, k', w)/\partial w = (b-1)\ln(b-1) - (b-2)\ln(b) - (b-2)/b.$$

For $7 \geq b \geq 4$ by direct calculation $(b-1)\ln(b-1) - (b-2)\ln(b) - (b-2)/b > 0$. In addition,

$$(b-1)\ln(b-1) - (b-2)\ln(b) > 1$$

is satisfied for $b \geq 7$. Note, $(b-1)\ln(b-1) - (b-2)\ln(b) = 0$ for $b = 2$ and increases for $b \geq 2$.

So, for $c = kb - k$ we can assume that $w = 0$, which corresponds to $b > k' = k$. Consequently, g_1, g_2 become

$$g_1(b, u, k) = b(b+u)(b-1)^k(b+u+k)^{b-2} \tag{15}$$

and

$$g_2(b, u, k) = b^k(b-k)(b+u+k-1)^k(b+u+k)^{b-k-1}. \tag{16}$$

Define $g_4(b, k, u) = \ln(g_1(b, k, u)/g_2(b, k, u))$. Then

$$\begin{aligned}\partial g_4(b, u, k)/\partial u &= 1/(b+u) + (k-1)/(b+u+k) - k/(b+u+k-1)\\ &= (k^2-k)/((b+u)(b+u+k)(b+u+k-1)) > 0.\end{aligned}$$

So, we can assume $u = 0$ and focus on the proof of the following:

$$(b+k)^{k-1}(b-1)^k > (b+k-1)^k b^{k-2}(b-k). \tag{17}$$

Let $g_4(b, k) = \ln((b+k)^{k-1}(b-1)^k/((b+k-1)^k b^{k-2}(b-k)))$. Then

$$\partial g_4(b, k)/\partial k = \ln(b+k) + \ln(b-1) - \ln(b+k-1) - \ln(b) + (k-1)/(b+k) + 1/(b-k) - k/(b+k-1).$$

For $k = 2$, $(b+k)^{k-1}(b-1)^k = (b+k-1)^k b^{k-2}(b-k) + 4$. We verify that $\partial g_4(b, k)/\partial k > 0$ for $k = 2$, $b = 3$. We also verify that

$$\partial^2 g_4(b, k)/(\partial k \partial b) = (-24b^5 + 8b^4 + 60b^3 - 28b^2 - 16)/(b(b^2-4)^2(b^2-1)^2) < 0$$

for $k = 2$ and $b \geq 3$ based on the standard evaluation (left to the reader), and that $\partial g_4(b, k)/\partial k$ asymptotically converges to 0 as b approaches infinity for $k = 2$. This implies that $\partial g_4(b, k)/\partial k > 0$ for $k = 2$. Furthermore, by straightforward evaluation (we leave it here to the reader) we obtain

$$\begin{aligned}&\partial^2 g_4(b, k)/\partial k^2 =\\ &\quad ((6b^3k + 6b^2k^2 + 2b^2 + 2bk^3 + 2k^2 + 2k^4) - (2b^3 + 7b^2k + 4bk^2 + 3k^3))/((b^2-k^2)^2(b+k-1)^2) > 0\end{aligned}$$

for $b > k \geq 2$. Hence, $(b+k)^{k-1}(b-1)^k > (b+k-1)^k b^{k-2}(b-k)$ for $k \geq 2$. Consequently, we obtain $g_1(b, c, k, u) > g_2(b, c, k, u, r)$ for $c = bk - k$.

We now assume that $k \leq c \leq bk$. Let $g_5(b, c, k, u, r) = \ln(g_1(b, c, k, u)/g_2(b, c, k, u, r))$ — based on (8) and (9). So, $g_5(b, c, k, u, r) > 0$ for $c = k + 1$ and for $c = kb - k$. By examining $\partial g_5(b, c, k, u, r)/\partial c = 0$ we conclude that there are at most two extreme points between $c = k$ and $c = bk$. For given b, k, u we have $g_1(b, k, k, u) = g_2(b, k, k, u, r)$ and $g_1(b, bk, k, u) = g_2(b, bk, k, u, r)$. This means that $g_5(b, c, k, u, r) = 0$ for $c = k$ and for $c = bk$. So, there must be exactly one extreme point (maximum) for $k+1 \leq c \leq kb-k$, which means that $g_1(b, c, k, u) > g_2(b, c, k, u, r)$ for $k+1 \leq c \leq kb-k$. □

Lemma 4.3. *Let b, c, k, be given positive integers with $b \geq 3$, $k \geq 3$, and $kb - k \geq c \geq 2k - 1$. Let u be given nonnegative integer. Let $x_0 = b$, $x_{k+1} = 1$, and let $f(x_1, x_2, \ldots, x_k) = \prod_{i=0}^{k}(x_i(b+u+i)^{x_i - x_{i+1}})$. Let $f_3 = f(x_1, x_2, \ldots, x_k)$ if $b - 1 \geq x_1 = x_2 = \cdots = x_{k-1} = (c - x_k)/(k-1)$, and let $f_2 = f(x_1, x_2, \ldots, x_k)$ if $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r \geq 1$ and $x_{r+1} = x_{r+2} = \cdots = x_k = 1$, for $r \geq 1$. Then $f_3 > f_2$ is satisfied over the region*

$$P := \left\{ x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1 \right\}.$$

Proof. We define functions $g_3(b, c, k, u, x_k)$, $g_2(b, c, k, u, r)$ corresponding to f_3, f_2 respectively as follows:

$$g_3(b, c, k, u, x_k) = bx_r((c-x_k)/(k-1))^{k-1}(b+u)^{b-\frac{c-x_k}{k-1}}(b+u+k-1)^{\frac{c-x_k}{k-1}-x_k}(b+u+k)^{x_k-1} \tag{18}$$

and $g_2(b, c, k, u, r)$ defined by (9) from Lemma 4.2.

For the purpose of this evaluation we assume $b, c, k, u, r \in R$. The proof follows by direct comparison of $g_3(b, c, k, u, x_k)$ with $g_2(b, c, k, u, r)$.

Define $h_3(b, c, k, u, x_k) = \ln(g_3(b, c, k, u, x_k)/g_3(b, c, k, u, 1))$. Then

$$\partial h_3(b, c, k, u, x_k)/\partial x_k = \\ 1/x_k - (k-1)/(c-x_k) + \ln(b+u+k) + (\ln(b+u))/(k-1) - (k/(k-1))\ln(b+u+k-1).$$

We note that $\frac{1}{x_k} - \frac{k-1}{c-x_k} \geq 0$, because by definition $x_k \leq x_{k-1} = \frac{c-x_k}{k-1}$. Then we verify that $\ln(b+u+k) + (\ln(b+u))/(k-1) - (k/(k-1))\ln(b+u+k-1) > 0$, for $b \geq 3$, $u \geq 0$ and $k \geq 3$. So, if $x_k = 1$ is feasible for given b, c, k then we can assume $x_k = 1$ for comparison of $g_3(b, c, k, u, x_k)$ with $g_2(b, c, k, u, r)$ (the worst case).

We first compare $g_3(b, c, k, u, x_k)$ with $g_2(b, c, k, u, r)$ for $c = 2k - 1$ (least possible) and for given b, k, u. Clearly, $x_k = 1$ is feasible in this case. So we assume $x_k = 1$. Suppose $g_3(b, c, k, u, x_k = 1) \leq g_2(b, c, k, u, r)$. Then, by Lemma 4.2 $f_1 = f(x_1, x_2, \ldots, x_{k-1}) > f(x_1, x_2, \ldots, x_{k-1}) = f_2$. This in turn implies $f_3 = f(x_1, x_2, \ldots, x_k) > f(x_1, x_2, \ldots, x_k) = f_2$ — a contradiction. So, $g_3(b, c, k, u, x_k = 1) > g_2(b, c, k, u, r)$ for $c = 2k - 1$. For $c = kb - k$ (largest possible), $g_3(b, c, k, u, x_k) > g_2(b, c, k, u, r)$ is directly implied by Lemma 4.2 because f_1 from Lemma 4.2 equals f_3.

We now assume that $k \leq c \leq bk$, and that $b \geq x_1 = x_2 = \cdots = x_{k-1} = (c - x_k)/k$ for $c > bk - k$. Let $g_5(b, c, k, u, x_k, r) = \ln(g_3(b, c, k, u, x_k)/g_2(b, c, k, u, r))$ — based on (18) and (9). So, $g_5(b, c, k, u, x_k, r) > 0$ for $c = 2k-1$ and for $c = kb-k$. By examining $\partial g_5(b, c, k, u, x_k, r)/\partial c = 0$ we conclude that there are at most two extreme points between $c = k$ and $c = bk$. For given b, k, u we have $g_3(b, k, k, u, 1) = g_2(b, k, k, u, r)$ and $g_3(b, bk, k, u, k) = g_2(b, bk, k, u, r)$. This means that $g_5(b, c, k, u, x_k, r) = 0$ for $c = k$ and for $c = bk$. So, there must be exactly one extreme point (maximum) for $2k - 1 \leq c \leq kb - k$, which means that $g_3(b, c, k, u, x_k) > g_2(b, c, k, u, r)$ for $2k - 1 \leq c \leq kb - k$. □

Lemma 4.4. *Let b, c, k, be given positive integers with $b \geq 3$ and $kb - k \geq c > k$. Let $x_0 = b$, $x_{k+1} = 1$, and let $g(x_1, x_2, \ldots, x_k) = \prod_{i=0}^{k}(x_i(b+i)^{x_i - x_{i+1}})$. The minimum of g over the region*

$$P := \left\{ x \in \mathbb{N}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq 1 \right\}$$

occurs at some point $(x_1, x_2, \ldots, x_k)$ if and only if $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r > 1$ and $x_{r+1} = x_{r+2} = \cdots = x_k = 1$, for some $r \geq 1$.

Proof. Suppose a minimum of g occurs at some point $(x_1, x_2, \ldots, x_k)$ where $b > x_i \geq x_{i+1} > 1$ is satisfied. Let r be the largest index for which $x_r > 1$. Then we have three cases to consider.

Case 1: $x_{r-1} = x_r$ is satisfied.

Let p be an index such that $x_{p-1} > x_p = x_{p+1} = \cdots = x_r$. Consider corresponding function $f_1 = f(x_p, x_{p+1}, \ldots, x_r)$ from Lemma 4.2, where $b = x_{p-1}$, $x \in R^{r-p+1}$. By Lemma 4.2, $f_1 > f_2$ — a contradiction.

Case 2: $x_{r-2} > x_{r-1} > x_r$ is satisfied.

Consider corresponding function $f(x_{r-1}, x_r)$ from Lemma 4.1, where $b = x_{r-2}$, $x \in R^2$. By Lemma 4.1, f is not a minimizer. By Lemma 4.2 $x_{r-1} = x_r$ is not minimizer either. So, either $x_{r-2} = x_{r-1}$ or $x_r = 1$ must be satisfied — a contradiction.

Case 3: $x_{r-2} = x_{r-1} > x_r$ is satisfied.

Let p be an index such that $x_{p-1} > x_p = x_{p+1} = \cdots = x_{r-1}$. Consider corresponding function $f_3 = f(x_p, x_{p+1}, \ldots, x_r)$ from Lemma 4.3, where $b = x_{p-1}$, $x \in R^{r-p+1}$. By Lemma 4.3, $f_3 > f_2$ — a contradiction.

So, by contradiction of Cases 1–3, the minimum of g must occur at some point $(x_1, x_2, \ldots, x_k)$, where $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r > 1$ and $x_{r+1} = x_{r+2} = \cdots = x_k = 1$, for some $r \geq 1$. □

Let $L_{n,m}$ be a special case of H such that $L_{n,m} = H(n; d_1, 1, 1, \ldots, 1)$. In the final fourth step we now state the following result.

Theorem 4.5. *Let n and m be positive integers so that there is a connected simple graph on n vertices and m edges. Then, for any connected graph G with n vertices and m edges, $t(G) \geq t(L_{n,m})$.*

Proof. Suppose $t(G)$ is minimum. Then, by Theorem 2.2 G can be transformed to H with $t(H)$ minimum too. If $m = n-1$ then H is a tree and the case is trivial. Hence, without loss of generality consider only case for $m > n-1$. If $H = H(n; 1, \ldots, 1)$ then $H = L_{n,m}$. Otherwise, $\sum_{i=1}^{k} d_i > k$. So, by Theorem 3.1 and Lemma 4.4 H must be of the form $H(n; d_1, d_2, \ldots, d_i, \ldots, d_k) = H(n; n-k, \ldots, n-k, d_i, 1, \ldots, 1)$, where $k, i \geq 1$, and $n - k \geq d_i > 1$. If $i = 1$ then $H = L_{n,m}$. If $i = 2$ then $H = H(n; n-k, d_2, 1, \ldots, 1)$ is isomorphic to $H(n; d_1^1, 1, \ldots, 1) = L_{n,m}$, where $d_1^1 = d_2$. Suppose $i > 2$. In this case $H = H(n; n-k, \ldots, n-k, d_i, 1, \ldots, 1)$ and it is isomorphic to $H(n; n-k, \ldots, n-k, d_{i-1}^1, 1, \ldots, 1)$, where $d_{i-1}^1 = d_i$. Furthermore, $H(n; n-k, \ldots, n-k, d_{i-1}^1, 1, \ldots, 1)$ can be transformed to $H'(n; n-k+1, \ldots, n-k+1, d_j^2, 1, \ldots, 1)$ for some $j \leq i$, which is not isomorphic to $H(n; n-k, \ldots, n-k, d_{i-1}^1, 1, \ldots, 1)$, i.e., $H(n; n-k, \ldots, n-k, d_{i-1}^1, 1, \ldots, 1) \neq H'(n; n-k+1, \ldots, n-k+1, d_j^2, 1, \ldots, 1)$. So, by Lemma 4.4

$$t(H'(n; n-k+1, \ldots, n-k+1, d_j^2, 1, \ldots, 1)) < t(H(n; n-k, \ldots, n-k, d_{i-1}^1, 1, \ldots, 1)),$$

a contradiction. □

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